

Models for Chronology Selection

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(February 1, 2008)

Abstract

In this paper, we derive an expression for the grand canonical partition function for a fluid of hot, rotating massless scalar field particles in the Einstein universe. We consider the number of states with a given energy as one increases the angular momentum so that the fluid rotates with an increasing angular velocity. We find that at the critical value when the velocity of the particles furthest from the origin reaches the speed of light, the number of states tends to zero. We illustrate how one can also interpret this partition function as the effective action for a boosted scalar field configuration in the product of three dimensional de Sitter space and S^1 . In this case, we consider the number of states with a fixed linear momentum around the S^1 as the particles are given more and more boost momentum. At the critical point when the spacetime is about to develop closed timelike curves, the number of states again tends to zero. Thus it seems that quantum mechanics naturally enforces the chronology protection conjecture by superselecting the causality violating field configurations from the quantum mechanical phase space.

PACS Number(s):

Typeset using REVTeX

I. INTRODUCTION

It is generally believed that any attempt to introduce closed timelike curves (CTCs) into the universe will fall foul of the chronology protection conjecture [1], which states that the laws of Physics somehow conspire to prevent one from manufacturing time machines. Early calculations supporting this conjecture concentrated on the behaviour of the renormalised energy-momentum tensor $\langle T_{\mu\nu} \rangle$, which was shown to diverge at the Cauchy horizon in a number of causality violating spacetimes. A possible mechanism for enforcement of chronology protection was therefore proposed as the back reaction of this divergent energy-momentum on the spacetime geometry via the semi-classical Einstein equations. Of course, it was hoped that the back reaction would be sufficiently strong enough to prevent the formation of CTCs. However, Kim and Thorne speculated [3] that if a full quantum theory of gravity were available, then one might find that the divergences cut off at some appropriate invariant distance from the Cauchy horizon, thus allowing the CTCs to form. Further doubts were cast when $\langle T_{\mu\nu} \rangle$ was calculated for scalar fields in two spacetimes with non-compactly generated Cauchy horizons. Boulware [4] and Tanaka and Hiscock [9] both found that for sufficiently massive fields in Gott space and Grant space respectively, $\langle T_{\mu\nu} \rangle$ could remain regular at the Cauchy horizon. More recently, it has been shown that Hadamard states exist in Misner space (in 2 and 4 dimensions) for which $\langle T_{\mu\nu} \rangle$ vanishes everywhere [5–7]. Misner space has a compactly generated Cauchy horizon and is therefore subject to the strong theorems recently proved by Kay, Radzikowski and Wald [2]. Cramer and Kay [8] have applied these theorems to the Misner space example and showed that even if there was no divergence as the Cauchy horizon was approached, $\langle T_{\mu\nu} \rangle$ must necessarily be ill defined on the Cauchy horizon itself. They also argue for similar behaviour in the noncompactly generated cases, but the fact that the energy-momentum tensor fails to diverge shows that back reaction does not enforce chronology protection.

In this paper, we adopt a slightly different approach by focusing on the effective action of a massless scalar field in a number of acausal spacetimes. From a formal point of view, the ef-

fective action is the fundamental field-theoretic quantity, from which the energy-momentum tensor is derived as a functional derivative with respect to the metric, so one would hope that an analysis of this quantity would provide new insights into issues of chronology protection. The effective action plays an important role in the Euclidean approach to quantum field theory on acausal spacetimes [10]. This approach can be used if some Euclidean space has an appropriate Lorentzian causality violating analytic continuation. CTCs do not exist in Euclidean space, so one can define a field theory on the Euclidean section, and then analytically continue to obtain the results valid for the acausal spacetime. In this formalism, one defines path integrals of the form

$$Z = \int \mathcal{D}[\phi] \exp(-S[\phi]) \quad (1)$$

over field configurations ϕ . Our aim is to provide thermodynamic and quantum cosmological interpretations to expressions of this type in the presence of causality violations. The main obstacle to any such interpretation comes from the fact that in general, the effective action density $\ln Z(x)$ diverges to infinity at the polarised hypersurfaces of acausal spacetimes [6]. Thus, if one was to construct a no-boundary amplitude for some causality violating geometry, then it would appear that creation of the universe was overwhelmingly favoured, contrary to ones intuitive hope for a strong suppression. However, we will argue that the effective action in itself does not yield the correct amplitude for creation, and that the true amplitude does indeed show suppression of acausal geometries.

In section II, we introduce two multiply connected Euclidean spaces, from which one can obtain a variety of acausal Lorentzian spacetimes. When one tries to do physics in a typical multiply connected spacetime, it is generally easier to work in the simply connected universal covering space, where points are identified under the action of some discrete group of isometries. The first example considered, therefore, is flat Euclidean space with points identified under a combined rotation plus translation. By analytically continuing the rotation to complex values ($\alpha \rightarrow a = i\alpha$), one can obtain Grant's generalisation of Misner space [12], which is just flat Minkowski space with points identified under a combined boost in the

(x, t) plane and orthogonal translation in the y direction. This spacetime contains CTCs in the left and right wedges (because of the boost identification) and is the covering space of the Gott spacetime [11], which describes two infinitely long cosmic strings moving past each other at high velocity. One can also find other acausal analytic continuations from this Euclidean section. We illustrate how one obtains the ‘spinning cone’ spacetime [13] and the metric for hot flat space, rotating rigidly with angular velocity Ω . In this latter spacetime, the velocity of a co-rotating observer increases as one moves radially outward from the axis of rotation and there will be acausal effects beyond the critical radius where this velocity reaches the speed of light.

The second example that we introduce is a new model, given by the Euclidean metric on $R \times S^3$, also identified under a combined rotation and translation. The basic reason for introducing this model is to provide a compact Euclidean space which could, in principle, contribute to a no-boundary path integral. It should not be surprising that the analytic continuations of this model have a causal structure qualitatively similar to the flat space examples. Indeed, by allowing the radius of the sphere to tend to infinity, one regains the periodically identified flat Euclidean space. The acausal spacetime analogous to Grant space, obtained by analytically continuing the rotation to a boost, is the product of three-dimensional de Sitter space and the real line ($3dS \times R$), periodically identified under a combined boost and translation.

In section III, we introduce a massless scalar field into these two Euclidean models and calculate the renormalised energy-momentum tensor in each case. Not surprisingly, we find that all the components of $\langle T_{\mu\nu} \rangle$ diverge at the Cauchy horizon and polarised hypersurfaces in all of the acausal analytic continuations.

Section IV is devoted to a calculation of the contributions to the effective action which diverge when one analytically continues the parameters of the Euclidean Einstein universe. In [6], a divergent contribution was derived from the de Witt-Schwinger asymptotic expansion of the heat kernel $H(x, x', \tau)$ about $\tau = 0$ and here, this contribution is rederived by integrating the energy-momentum tensor. When one integrates $\langle T_{\mu\nu} \rangle$, however, one also

finds other divergent contributions that do not appear when one calculates $\ln Z(x)$ directly from the heat kernel expansion. Ultimately, though, the dominant divergence is the same as before – the effective action density diverges to infinity at the polarised hypersurfaces of the acausal analytic continuations.

In section V, we give a physical interpretation to the results of the previous section. Firstly, we consider a fluid of hot, rotating scalar particles in the Einstein universe. The grand canonical partition function for these particles is given by $\ln Z(x)$, with the parameter α continued to complex values so that the fluid is rotating with angular velocity $\Omega = i\frac{\alpha}{\beta}$, where β is the inverse temperature. Energy and angular momentum are conserved quantities in the Einstein universe, and we derive expressions for the energy of the particles and also the angular momentum that is required to make the fluid rotate with an angular velocity Ω . At a critical angular velocity $\Omega = \frac{1}{r}$, where r is the radius of the Einstein universe, these expressions diverge which means that one would have to inject an infinite amount of angular momentum into the system if one wanted the velocity of the particles to reach the speed of light. Ultimately, however, one is interested in the behaviour of the number of states with a given energy as the angular momentum is increased. In order to keep the energy fixed, one must decrease the temperature as more and more angular momentum is put in so that the angular velocity of the particles approaches its critical value. One finds that the entropy of the scalar particles diverges to minus infinity as the velocity of the particles approaches the speed of light (or $\Omega \rightarrow \frac{1}{r}$). Since the entropy is just the logarithm of the number of states, one can conclude that there are no quantum states available for speed of light rotation.

These results can be interpreted analogously if one analytically continues the Euclidean section to obtain $3dS \times S^1$, the product of three dimensional de Sitter space and the S^1 with length β . The conserved quantities are now linear and boost momentum so this time, one wants to consider the number of states with a fixed linear momentum as one gives the particles more and more boost momentum. There is a critical boost at which the spacetime will develop CTCs (when $a = \beta/r$), but the amount of boost momentum that is needed to achieve this is once again infinite and the entropy diverges to minus infinity at this critical

value. In this case, therefore, there are no quantum states available for these causality violating field configurations.

The no-boundary amplitude Ψ_m which describes the creation of the spacetime $3dS \times S^1$ from nothing is also constructed. Ψ_m^2 is the microcanonical partition function, or density of states, which tends to zero as one adjusts the parameters so that the spacetime is about to develop CTCs. The amplitude $\ln \Psi_m$ is obtained from the original effective action by a Legendre transform, just as one obtains the entropy from the partition function in a thermodynamic context. Thus the message of this paper is that it is possible to recover a sensible interpretation of Euclidean path integrals in the presence of causality violation as long as one focuses on the density of states. It seems highly likely that this quantity will always tend to zero as one tries to introduce CTCs, thus enforcing the chronology protection conjecture.

II. PERIODICALLY IDENTIFIED EUCLIDEAN SPACES

In this section, we illustrate how CTCs can be introduced into a spacetime by identifying points under the action of a discrete group of isometries. Consider the metric for flat Euclidean space in cylindrical polar coordinates

$$ds^2 = d\tau'^2 + dr'^2 + r'^2 d\phi'^2 + dz'^2 \quad (2)$$

where points are identified under a combined rotation and translation, *i.e.* (τ', r', ϕ', z') and $(\tau' + n\beta, r', \phi' + n\alpha, z')$ represent the same spacetime point (where n is some integer). The identification parameters appear explicitly in the metric when one makes the coordinate transformation

$$\begin{aligned} \tau &= \tau' - \frac{\beta\phi'}{\alpha} \\ r &= r' \\ \alpha\phi &= \phi' \\ z &= z' \end{aligned} \quad (3)$$

to obtain

$$ds^2 = (d\tau + \beta d\phi)^2 + dr^2 + \alpha^2 r^2 d\phi^2 + dz^2. \quad (4)$$

The idea now is to analytically continue one of the parameters α or β to obtain the acausal Lorentzian spacetimes. If we continue $\beta \rightarrow b = -i\beta$ and then set $t = -i\tau$, we obtain

$$ds^2 = -(dt + bd\phi)^2 + dr^2 + \alpha^2 r^2 d\chi^2 + dz^2 \quad (5)$$

which is the ‘spinning cone’ metric [13], the spacetime produced by an infinitely long string with angular momentum b . The condition for CTCs is $(-b^2 + \alpha^2 r^2) < 0$, so the causality violating region is just $0 < r < \frac{b}{\alpha}$. The spinning cone metric is singular along the axis of the string and will not concern us further. It is interesting, however, to note that this spacetime and Grant space (obtained in the next paragraph) are just different analytic continuations of the same Euclidean metric.

Returning to equation (4), if one now continues $\alpha \rightarrow a = i\alpha$, one obtains the metric

$$ds^2 = -a^2 r^2 d\phi^2 + dr^2 + (d\tau + \beta d\phi)^2 + dz^2. \quad (6)$$

Analytically continuing in α means that points are now identified under a combined boost plus translation, so this metric is just that of Grant’s generalised Misner space. The condition for CTCs is $(\beta^2 - a^2 r^2) < 0$. In other words, the CTCs inhabit the region where $r > \frac{\beta}{a}$. It should be stressed that the surface defined by $r = \frac{\beta}{a}$ is not the Cauchy horizon for Grant space. If one thinks of the (t, x) section of Minkowski space as being divided up into the usual four wedges, then for Grant space the CTCs are confined to the $r > \frac{\beta}{a}$ region of the left and right wedges but the Cauchy horizon is defined by $t = \pm x$. The Cauchy horizon is in fact the $n \rightarrow \infty$ limiting surface of a family of n th polarised hypersurfaces. Physically, the n th polarised hypersurface is defined as the set of points which can be joined to themselves by a (self-intersecting) null geodesic which loops around the space n times. In Grant space, these surfaces are defined by the equation

$$2r^2(1 - \cosh(na)) + n^2\beta^2 = 0. \quad (7)$$

In the limit as $n \rightarrow 0$, one obtains $r = \frac{\beta}{a}$, so one could say that this surface is the zeroth polarised hypersurface but in light of the above definition, its physical interpretation is unclear. Bearing this in mind however, we shall continue to refer to it as the zeroth polarised hypersurface. Ordinary Misner space is obtained when the translation parameter β is zero. Misner space is basically the Euclidean cosmic string metric, with the angular deficit parameter continued to complex values.

A more familiar example of a spacetime containing CTCs is obtained from the original metric (2) by the coordinate transformation

$$\begin{aligned}\beta\tau &= \tau' \\ r &= r' \\ \phi &= \phi' - \frac{\alpha\tau'}{\beta} \\ z &= z'.\end{aligned}\tag{8}$$

If one analytically continues $\beta \rightarrow b = -i\beta$ in this case, one obtains the metric for hot flat space, rotating rigidly with angular velocity $\Omega = \frac{a}{b} = \frac{a}{\beta}$

$$ds^2 = -b^2 d\tau^2 + dr^2 + r^2(d\phi + \alpha d\tau)^2 + dz^2.\tag{9}$$

In this metric, the Killing vector $\partial/\partial\tau$ becomes spacelike beyond the critical radius where the velocity of a co-rotating observer exceeds the speed of light.

In a later section, we will be concerned with the possible contributions from acausal metrics to no-boundary amplitudes. The flat space examples above could not contribute because their Euclidean section is noncompact. However, one can readily construct a compact example with spherical spatial sections, given by the Euclidean metric on $R \times S^3$ (the Euclidean Einstein universe)

$$ds^2 = d\tau^2 + r^2(d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2))\tag{10}$$

where the points $(\tau, \chi, \theta, \phi)$ and $(\tau + m\beta, \chi, \theta, \phi + m\alpha)$ are identified. Clearly one can analytically continue the metric parameters to obtain acausal spacetimes analogous to the flat

space examples considered above. The spacetime obtained by just analytically continuing $\alpha \rightarrow a = i\alpha$ is the product of three-dimensional de Sitter space and the real line, periodically identified under a combined boost and translation, and in this case the polarised hypersurfaces are defined by the equation

$$\sin^2 \chi \sin^2 \theta = \frac{1 - \cosh\left(\frac{m\beta}{r}\right)}{1 - \cosh(ma)}. \quad (11)$$

The polarised hypersurfaces all coincide at the critical value when $a = \beta/r$ and $\sin \chi \sin \theta = 1$, and CTCs appear in the spacetime if a is increased further. By taking the radius r of the sphere to infinity, one obtains the flat space example as a limiting case.

III. SCALAR FIELD ENERGY-MOMENTUM TENSOR

Now consider placing a massless scalar field on the two identified Euclidean spaces described in the previous section. To find the energy-momentum for either of these spaces, one just applies the standard second order differential operator to the appropriate Euclidean Green function. Analytically continuing at the end of the calculation will yield the results for the acausal Lorentzian spacetimes. We first consider the flat space example.

The renormalised Euclidean Green function for a massless scalar field on identified flat space is written using the method of images as

$$D(x, x') = \frac{1}{4\pi^2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{\sigma_n(x, x')}, \quad (12)$$

where

$$\sigma_n(x, x') = r^2 + r'^2 - 2rr' \cos(\phi - \phi' - n\alpha) + (\tau - \tau' - n\beta)^2 + (z - z')^2. \quad (13)$$

The energy-momentum tensor is obtained by differentiating D according to

$$\langle T_{\mu\nu} \rangle = \lim_{x' \rightarrow x} \left[\frac{2}{3} D_{;\nu'\mu} - \frac{1}{3} D_{;\nu\mu} - \frac{1}{6} g_{\mu\nu} D^{;\sigma'\sigma} \right] \quad (14)$$

and the individual components are given by

$$\langle T_{\tau\tau} \rangle = \frac{1}{4\pi^2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{2(\cos(n\alpha) + 2)}{3\sigma_n(x, x)^2} - \frac{4n^2\beta^2(\cos(n\alpha) + 5)}{3\sigma_n(x, x)^3} \quad (15)$$

$$\langle T_{rr} \rangle = \frac{1}{4\pi^2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{2(\cos(n\alpha) + 2)}{3\sigma_n(x, x)^2} \quad (16)$$

$$\langle T_{\phi\phi} \rangle = \frac{1}{4\pi^2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{2r^2(\cos(n\alpha) + 2)}{3\sigma_n(x, x)^2} \left[-3 + \frac{4n^2\beta^2}{\sigma_n(x, x)} \right] \quad (17)$$

$$\langle T_{zz} \rangle = \frac{1}{4\pi^2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{2(\cos(n\alpha) + 2)}{3\sigma_n(x, x)^2} - \frac{4n^2\beta^2(\cos(n\alpha) - 1)}{3\sigma_n(x, x)^3} \quad (18)$$

$$\langle T_{\tau\phi} \rangle = \frac{1}{4\pi^2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} -\frac{8n\beta r^2 \sin(n\alpha)}{\sigma_n(x, x)^3} \quad (19)$$

These results are valid on the Euclidean section. One obtains the energy-momentum components for generalised Misner space (reproducing Grant's results) by analytically continuing $\alpha \rightarrow a = i\alpha$. We also note that continuing in β (and τ) yields the energy-momentum for the spinning cone.

The appropriate Green function for a massless scalar field on $R \times S^3$ identified under a combined rotation and translation is

$$D(x, x') = \frac{1}{4\pi^2 r} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{(s_m + 2\pi nr)}{\sin\left(\frac{s_m}{r}\right)} \frac{1}{\lambda_{mn}(x, x')} \quad (20)$$

where $\lambda_{m\pm n}(x, x') = (\tau - \tau' - m\beta)^2 + (s_m \pm 2\pi nr)^2$ and $s_m = r \cos^{-1}(\cos \chi \cos \chi' + \sin \chi \sin \chi' (\cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi' - m\alpha)))$. By combining terms of positive and negative n , one can write $D(x, x')$ as the series

$$\frac{1}{4\pi^2 r} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{s_m}{\sin\left(\frac{s_m}{r}\right)} \frac{f_{mn}}{\lambda_{mn}\lambda_{m-n}} \quad (21)$$

where $f_{mn} = (\tau - \tau' - m\beta)^2 + (s_m + 2\pi nr)(s_m - 2\pi nr)$. The Green function is renormalised by dropping the $n = 0, m = 0$ term in the sum, as this term is the only divergent one as

the points are brought together. It is also convenient to separate the Green function as $D = D_1 + D_2$, where D_1 is the $m = 0, \sum_n$ part of the Green function. In the limit as $x' \rightarrow x$, D_1 is given by [14]

$$\lim_{x' \rightarrow x} D_1 = -\frac{1}{48\pi^2 r^2}. \quad (22)$$

This is just the value for the Einstein universe without identifications. The remainder of the Green function can be written as

$$D_2 = \frac{1}{4\pi^2 r} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \sum_{n=-\infty}^{\infty} \frac{s_m}{16\pi^4 r^4 \sin\left(\frac{s_m}{r}\right)} \frac{f_{mn}}{(n+z_1)(n-z_1)(n+z_1^*)(n-z_1^*)} \quad (23)$$

where the complex quantity

$$z_1 = \frac{s_m + i(\tau - \tau' - m\beta)}{2\pi r}. \quad (24)$$

The sum over n can be evaluated using the method of residues. We find that

$$\begin{aligned} D_2 &= \frac{1}{16\pi^2 r^2} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{\cot(\pi z_1) + \cot(\pi z_1^*)}{\sin\left(\frac{s_m}{r}\right)} \\ &= \frac{1}{16\pi^2 r^2} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{1}{\sin\left(\frac{s_m + i(\tau - \tau' - m\beta)}{2r}\right) \sin\left(\frac{s_m - i(\tau - \tau' - m\beta)}{2r}\right)} \\ &= \frac{1}{8\pi^2 r^2} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{1}{\cosh\left(\frac{\tau - \tau' - m\beta}{r}\right) - \cos\left(\frac{s_m}{r}\right)} \end{aligned} \quad (25)$$

which represents D_2 as a sum over ordinary Einstein Green functions, as one might expect.

Furthermore, this quantity has already been renormalised, so all that remains is to apply the standard formula to calculate the energy-momentum

$$\langle T_{\mu\nu} \rangle = \lim_{x' \rightarrow x} \left(\frac{2}{3} D_{;\nu'\mu} - \frac{1}{3} D_{;\nu\mu} - \frac{1}{6} g_{\mu\nu} D^{;\sigma'}_{\sigma} + \frac{1}{3} g_{\mu\nu} D^{;\sigma'}_{\sigma'} + \frac{1}{6} (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) D \right). \quad (26)$$

The individual components are

$$\begin{aligned} \langle T_{\tau\tau} \rangle &= -\frac{1}{480\pi^2 r^4} + \frac{1}{24\pi^2 r^2} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \left\{ \frac{\cosh\left(\frac{m\beta}{r}\right) + \cos(m\alpha) + 1}{r^2 \sigma_m(x)^2} \right. \\ &\quad \left. + \frac{2(1 + \cos(m\alpha)) \left(1 - \cosh\left(\frac{m\beta}{r}\right)\right) - 4 \sinh^2\left(\frac{m\beta}{r}\right)}{r^2 \sigma_m(x)^3} \right\} \end{aligned} \quad (27)$$

$$\langle T_{\chi\chi} \rangle = \frac{1}{1440\pi^2 r^2} + \frac{1}{24\pi^2 r^2} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \left\{ \frac{\cosh\left(\frac{m\beta}{r}\right) + \cos(m\alpha) + 1}{\sigma_m(x)^2} - \frac{2 \cos^2 \theta (1 - \cos(m\alpha)) \left(1 - \cosh\left(\frac{m\beta}{r}\right)\right)}{\sigma_m(x)^3} \right\} \quad (28)$$

$$\langle T_{\chi\theta} \rangle = \frac{1}{24\pi^2 r^2} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{2 \sin \chi \cos \chi \sin \theta \cos \theta (1 - \cos(m\alpha)) \left(1 - \cosh\left(\frac{m\beta}{r}\right)\right)}{\sigma_m(x)^3} \quad (29)$$

$$\langle T_{\tau\phi} \rangle = -\frac{1}{8\pi^2 r^2} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{2 \sin^2 \chi \sin^2 \theta \sinh\left(\frac{m\beta}{r}\right) \sin(m\alpha)}{r \sigma_m(x)^3} \quad (30)$$

$$\begin{aligned} \langle T_{\theta\theta} \rangle = & \frac{\sin^2 \chi}{1440\pi^2 r^2} + \frac{\sin^2 \chi}{24\pi^2 r^2} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \left\{ \frac{3 - \cosh\left(\frac{m\beta}{r}\right) + \cos(m\alpha)}{\sigma_m(x)^2} \right. \\ & \left. + \frac{2 \left(1 - \cosh\left(\frac{m\beta}{r}\right)\right)^2 - 2 \sin^2 \theta (1 - \cos(m\alpha)) \left(1 - \cosh\left(\frac{m\beta}{r}\right)\right)}{\sigma_m(x)^3} \right\} \end{aligned} \quad (31)$$

$$\begin{aligned} \langle T_{\phi\phi} \rangle = & \frac{\sin^2 \chi \sin^2 \theta}{1440\pi^2 r^2} + \frac{\sin^2 \chi \sin^2 \theta}{24\pi^2 r^2} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \left\{ \frac{-\left(\cosh\left(\frac{m\beta}{r}\right) + 3 \cos(m\alpha) + 5\right)}{\sigma_m(x)^2} \right. \\ & \left. + \frac{2 \sinh^2\left(\frac{m\beta}{r}\right) - 4 \left(1 + \cos(m\alpha)\right) \left(1 - \cosh\left(\frac{m\beta}{r}\right)\right)}{\sigma_m(x)^3} \right\} \end{aligned} \quad (32)$$

where $\sigma_m(x) = \cosh\left(\frac{m\beta}{r}\right) - 1 + \sin^2 \chi \sin^2 \theta (1 - \cos(m\alpha))$. Clearly all of these components diverge at the n th polarised hypersurfaces if one analytically continues $\alpha \rightarrow a = i\alpha$.

IV. DIVERGENT CONTRIBUTIONS TO THE EFFECTIVE ACTION

As we stated in the introduction, the ultimate aim of this paper is to provide sensible interpretations for path integrals of the form

$$Z = \int \mathcal{D}[\phi] \exp(-S[\phi]) \quad (33)$$

in the presence of causality violations. The trouble is that if one calculates the effective action density $\ln Z(x)$ for matter fields in an acausal spacetime, then the results of [6] suggest that $\ln Z(x)$ generally diverges to infinity at each of the n th polarised hypersurfaces as one analytically continues the background so that the Lorentzian section is about to develop CTCs.

For example, consider the periodically identified Euclidean Einstein universe. From the energy-momentum tensor calculated at the end of the previous section, one can define the change in effective action induced by a metric perturbation $\delta g_{\mu\nu}$ as

$$\delta \ln Z = \frac{1}{2} \int g^{\frac{1}{2}} \langle T^{\mu\nu} \rangle \delta g_{\mu\nu} d^4x. \quad (34)$$

In this case, the metric perturbations arise by varying the parameter α , so that if one begins with the metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = d\tau^2 + r^2 \left(d\chi^2 + \sin^2 \chi \left(d\theta^2 + \sin^2 \theta \left(d\phi + \frac{\alpha}{\beta} d\tau \right)^2 \right) \right), \quad (35)$$

then the perturbed metric $g_{\mu'\nu'} = g_{\mu\nu} + \delta g_{\mu\nu}$ is obtained from the original one by the coordinate transformation $\phi = \phi' + \tau' \frac{d\alpha}{\beta}$. The only nonzero perturbations are $\delta g_{\tau\phi} = r^2 \sin^2 \chi \sin^2 \theta \frac{d\alpha}{\beta}$ and $\delta g_{\tau\tau} = 2r^2 \sin^2 \chi \sin^2 \theta \frac{\alpha d\alpha}{\beta^2}$, which implies that the total change in action is given by

$$\delta \ln Z = \int g^{\frac{1}{2}} \left(\langle T_{\tau\phi} \rangle - \frac{\alpha}{\beta} \langle T_{\phi\phi} \rangle \right) \frac{d\alpha}{\beta} d^4x. \quad (36)$$

Integrating up with respect to α should yield the total effective action. The first term contributes

$$\frac{1}{4\pi^2 r^4} \sum_{m=1}^{\infty} \frac{\left(\frac{m\beta}{r} \right)^{-1} \sinh \left(\frac{m\beta}{r} \right)}{\sigma_m(x)^2} \quad (37)$$

to the effective Lagrangian, which diverges at the polarised hypersurfaces when one analytically continues $\alpha \rightarrow a = i\alpha$ to obtain $3dS \times R$, the product of three-dimensional de Sitter space and the real line, periodically identified under a combined boost and translation. One can take the $r \rightarrow \infty$ limit to obtain the contribution to the effective Lagrangian in Grant space (if one defines a new radial coordinate $r' = r \sin \chi \sin \theta$)

$$\frac{1}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{\left(2r'^2(1 - \cosh(ma)) + m^2\beta^2\right)^2}. \quad (38)$$

We note that the expressions obtained here agree with those of [6], obtained by a different method. However, (36) indicates that there should be an additional contribution to the effective Lagrangian, given by $I = -\int \langle T_{\phi\phi} \rangle \alpha \frac{d\alpha}{\beta^2}$. This can be integrated by parts to obtain

$$I = -\frac{1}{\beta^2} \left\{ \alpha \int \langle T_{\phi\phi} \rangle d\alpha - \int \left(\int \langle T_{\phi\phi} \rangle d\alpha \right) d\alpha \right\}. \quad (39)$$

The relevant integrals can be solved by successive application of the formulae [15]

$$\begin{aligned} \int \frac{A + B \cos x}{(a + b \cos x)^n} dx &= \frac{1}{(n-1)(a^2 - b^2)} \int \frac{(n-1)(Aa - Bb) - (n-2)(Ab - Ba) \cos x}{(a + b \cos x)^{n-1}} dx \\ &\quad - \frac{(Ab - Ba) \sin x}{(n-1)(a^2 - b^2)(a + b \cos x)^{n-1}} \end{aligned} \quad (40)$$

and

$$\int \frac{dx}{a + b \cos x} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \left\{ \frac{(a-b) \tan \frac{x}{2}}{\sqrt{a^2 - b^2}} \right\}. \quad (41)$$

One finds that the divergent contribution to the effective Lagrangian is given by

$$\begin{aligned} &\frac{\sin^2 \chi \sin^2 \theta}{24\pi^2 r^4} \sum_{m=1}^{\infty} \left(\frac{m\beta}{r} \right)^{-2} \left\{ \frac{g(\beta)ma \sinh(ma)}{h(\beta)\sigma_m(x)^2} + \frac{X(\beta)ma \sinh(ma)}{\left(\cosh\left(\frac{m\beta}{r}\right) - 1\right) h(\beta)^2 \sigma_m(x)} \right. \\ &\quad \left. - \frac{g(\beta)}{\sin^2 \chi \sin^2 \theta h(\beta)\sigma_m(x)} + \frac{X(\beta) \ln \sigma_m(x)}{\sin^2 \chi \sin^2 \theta \left(\cosh\left(\frac{m\beta}{r}\right) - 1\right) h(\beta)^2} \right\} \end{aligned} \quad (42)$$

where

$$g(\beta) = 4 \left(\cosh\left(\frac{m\beta}{r}\right) - 1 \right) + 8 \sin^2 \chi \sin^2 \theta + 2 \sin^2 \chi \sin^2 \theta \left(\cosh\left(\frac{m\beta}{r}\right) + 1 \right) \quad (43)$$

$$h(\beta) = \cosh\left(\frac{m\beta}{r}\right) - 1 + 2 \sin^2 \chi \sin^2 \theta \quad (44)$$

$$\begin{aligned} X(\beta) &= (g(\beta) - 6h(\beta)) \left(\cosh\left(\frac{m\beta}{r}\right) - 1 + \sin^2 \chi \sin^2 \theta \right) + 2 \sin^2 \chi \sin^2 \theta \left(g(\beta) \right. \\ &\quad \left. - h(\beta) \left(\cosh\left(\frac{m\beta}{r}\right) + 5 \right) + 2 \left(\cosh\left(\frac{m\beta}{r}\right) - 1 \right) \left(\cosh\left(\frac{m\beta}{r}\right) + 1 \right) \right) \end{aligned} \quad (45)$$

Ultimately, one is interested in the most dominant divergence in the effective Lagrangian at the polarised hypersurfaces of the acausal analytic continuation. For our purposes, therefore, one only wants the terms that diverge at least as strongly as $\frac{1}{\sigma_m(x)^2}$, which is how (37) behaves. All other terms, including the finite contributions, can be neglected in future calculations without losing any of the essential physics. Finally, therefore, one obtains

$$\ln Z(x) = \frac{1}{4\pi^2 r^4} \sum_{m=1}^{\infty} \left\{ \frac{\left(\frac{m\beta}{r}\right)^{-1} \sinh\left(\frac{m\beta}{r}\right)}{\sigma_m(x)^2} + \frac{\left(\frac{m\beta}{r}\right)^{-2} g(\beta) m a \sinh(ma)}{6h(\beta)\sigma_m(x)^2} \right\} \quad (46)$$

as the dominant contribution. The first point to note about the second term in this expression is that it reinforces the first term, *i.e.* it also diverges to infinity at each of the polarised hypersurfaces. In fact, both terms are equal at the point where all the polarised hypersurfaces coincide (*i.e.* if $a = \frac{\beta}{r}$ and $\sin \chi \sin \theta = 1$). However, unlike (37), the second term cannot be derived from the asymptotic expansion of the heat kernel $H(x, x', \tau)$ near $\tau = 0$.

V. THE SUPPRESSION OF ACAUSAL EFFECTS

Now let us consider the Einstein universe as a fixed background on which scalar particles can exist. In this universe, one defines energy and angular momentum by integrating the energy-momentum tensor with the appropriate Killing vector over a spacelike surface and these quantities are conserved in that they are the same on all surfaces. If one now puts a certain energy in scalar particles in this universe, it will occupy a number of states given by the entropy, and if the particles are given angular momentum, the fluid will begin to rotate and the number of states will decrease.

The effective action density $\ln Z(x)$ for the scalar particles is given by the expression (46), equally valid for both analytic continuations of the Euclidean section. Here, one can interpret $\ln Z(x)$ as the grand canonical partition function for the hot rotating radiation. The $r \rightarrow \infty$ limit gives the partition function for rotating scalar radiation in flat space, and if the angular velocity parameter a is small, one obtains the partition function for a hot

rigidly rotating perfect scalar fluid (when one integrates $\ln Z(x)$ over a cylindrical volume with radius $r' = r_B$)

$$\ln Z = \frac{\pi^2 V}{90\beta^3} \frac{1}{\left(1 - \left(\frac{ar_B}{\beta}\right)^2\right)}. \quad (47)$$

The partition function satisfies

$$\ln Z = \mathcal{S} - \beta(E - \Omega J), \quad (48)$$

and by applying the standard thermodynamic identities, one can now calculate the energy of the particles at a temperature $T = \beta^{-1}$ and also the angular momentum that is required to make the particles rotate with an angular velocity $\Omega = a/\beta$. In the Einstein universe,

$$\begin{aligned} E(x) = -\frac{\partial \ln Z(x)}{\partial \beta} = & \sum_{m=1}^{\infty} \frac{2 \ln Z_m(x)}{\sigma_m(x)} \left(\frac{\partial \sigma_m(x)}{\partial \beta} \right) + \frac{1}{4\pi^2 r^4} \sum_{m=1}^{\infty} \frac{1}{\beta \sigma_m(x)^2} \\ & \left\{ \left(\frac{m\beta}{r} \right)^{-1} \sinh \left(\frac{m\beta}{r} \right) - \cosh \left(\frac{m\beta}{r} \right) + \frac{\sin^2 \chi \sin^2 \theta}{6} \left(\frac{m\beta}{r} \right)^{-2} \frac{ma \sinh(ma)}{h(\beta)^2} \right. \\ & \left. (2g(\beta)h(\beta) - \beta g'(\beta)h(\beta) + \beta g(\beta)h'(\beta)) \right\} \end{aligned} \quad (49)$$

$$\begin{aligned} J(x) = \frac{\partial \ln Z(x)}{\partial a} = & -\sum_{m=1}^{\infty} \frac{2 \ln Z_m(x)}{\sigma_m(x)} \left(\frac{\partial \sigma_m(x)}{\partial a} \right) + \frac{1}{4\pi^2 r^4} \sum_{m=1}^{\infty} \frac{1}{a \sigma_m(x)^2} \\ & \left\{ \frac{\sin^2 \chi \sin^2 \theta}{6} \left(\frac{m\beta}{r} \right)^{-2} \frac{g(\beta)}{h(\beta)} (ma \sinh(ma) + (ma)^2 \cosh(ma)) \right\} \end{aligned} \quad (50)$$

where $\ln Z(x) = \sum_{m=1}^{\infty} \ln Z_m(x)$. These expressions diverge to infinity at the critical angular velocity $\Omega = 1/r$ (if $\sin \chi \sin \theta = 1$). Physically, this means that one would have to put an infinite amount of angular momentum into the system if one wanted the particles at the boundary to move at the speed of light. If one now calculates the entropy of the particles, then one obtains

$$\begin{aligned} \mathcal{S}(x) = & \sum_{m=1}^{\infty} \frac{2 \ln Z_m(x)}{\sigma_m(x)} \left[\left(\frac{m\beta}{r} \right) \sinh \left(\frac{m\beta}{r} \right) - \sin^2 \chi \sin^2 \theta ma \sinh(ma) \right] + \frac{1}{4\pi^2 r^4} \sum_{m=1}^{\infty} \frac{1}{\sigma_m(x)^2} \\ & \left\{ 2 \left(\frac{m\beta}{r} \right)^{-1} \sinh \left(\frac{m\beta}{r} \right) - \cosh \left(\frac{m\beta}{r} \right) - \frac{\sin^2 \chi \sin^2 \theta}{6} \left(\frac{m\beta}{r} \right)^{-2} \frac{g(\beta)}{h(\beta)} (ma)^2 \cosh(ma) \right. \\ & \left. + \frac{\sin^2 \chi \sin^2 \theta}{6} \left(\frac{m\beta}{r} \right)^{-2} \frac{ma \sinh(ma)}{h(\beta)^2} (2g(\beta)h(\beta) - \beta g'(\beta)h(\beta) + \beta g(\beta)h'(\beta)) \right\} \end{aligned} \quad (51)$$

We want to consider what happens to the number of states with a given energy as the angular momentum is increased. However, as one injects more and more angular momentum into the system so that the angular velocity approaches the critical value $\Omega = 1/r$, the energy of the particles also diverges to infinity. This means that in order to keep the energy fixed, one must decrease the fluid temperature by an appropriate amount as the energy increases. In particular, as the energy diverges to infinity, the temperature must be scaled to zero. Therefore, as Ω approaches its critical value and as the parameter β tends to infinity, one can see that the entropy diverges to minus infinity. This means that as one gets nearer and nearer to making the particles travel faster than the speed of light, their number of states decreases to zero.

Of course, the identified Euclidean Einstein universe can be analytically continued in a different way to obtain $3dS \times S^1$, the product of three dimensional de Sitter space and the S^1 with length β . In this case, however, the conserved quantities are no longer energy and angular momentum, but rather linear momentum and boost momentum. Once again, therefore, one can consider $3dS \times S^1$ as a fixed background containing scalar field particles with a fixed amount of linear momentum, occupying a certain number of states. One can give these particles boost momentum and the amount that is needed to boost the particles to a certain value of a is again determined by the formula (50). The number of states available for the particles is given by the entropy, which decreases as the particles are boosted to higher and higher values. As was discussed in section II, CTCs appear in this spacetime at the critical value $a = \beta/r$, but the amount of boost momentum that is needed to obtain this critical value is actually infinite, and the corresponding number of states available to the system falls to zero as \mathcal{S} diverges to minus infinity. In this example, therefore, one can see that there are insuperable obstacles to introducing CTCs and no available quantum states for causality violation.

Finally, let us consider the creation of $3dS \times S^1$ from nothing, a process that can be described by constructing a no-boundary wave function. Specifically, we want the amplitude Ψ to propagate from nothing to a boosted scalar field configuration on the three-surface with

topology $S^1 \times S^2$ at constant ϕ . One must have a fixed linear momentum around the S^1 , characterised by the parameter β , while the amount of boost is determined by the parameter a . The wave function will be given by a Euclidean path integral of the usual form, but once again care must be taken. The amplitude is described by cutting the original solution in half, but it would be a mistake to simply use the amplitude $\ln \Psi = \ln Z/2$. One can see that this would be tantamount to employing a grand canonical description, giving the amplitude as a function of the fixed ‘potentials’ a and β but as we have stressed, the correct amplitude should be given as a function of the conserved ‘charges’ appropriate for a microcanonical description. The correct amplitude Ψ_m is defined as the Legendre transform

$$2 \ln \Psi_m = \ln Z - \beta \frac{\partial \ln Z}{\partial \beta} - a \frac{\partial \ln Z}{\partial a}. \quad (52)$$

The microcanonical partition function, or density of states, is given by the quantity Ψ_m^2 , which implies that the entropy is just $2 \ln \Psi_m$. In this example, therefore, one can see that the amplitude to propagate from nothing to a boosted scalar field configuration is nonzero if the boost is not too large. As soon as it becomes large enough to lead to the formation of CTCs, however, the amplitude vanishes exponentially.

VI. DISCUSSION

In this paper, we have considered the behaviour of a scalar quantum field on a background spacetime whose metric parameters can be adjusted so as to introduce CTCs. The entropy has been shown to diverge to minus infinity at the onset of causality violation, which can be interpreted as saying that the number of available quantum states tends to zero. The crucial question to ask, therefore, is whether this result holds in the general case.

The key quantity in our analysis has been the effective action density, which initially diverges to infinity. In section IV, it was shown that the strongest divergence in this action has two distinct contributions. The first contribution can be derived from the asymptotic expansion of the heat kernel, and it has been shown that in general this contribution diverges

to infinity for fields of arbitrary mass and spin at the polarised hypersurfaces of an acausal spacetime [6]. The only exceptions to this rule occur if the Van–Vleck determinant is made to vanish, as in Visser’s Roman ring configuration [16]. However, the second contribution to the action cannot be derived from a knowledge of ultra–violet behaviour and need not depend on the Van–Vleck determinant. One would expect this term to diverge with the same sign as the other dominant contribution, so even in the case of the Roman ring the action should still diverge to infinity, although this remains to be explicitly shown.

Many chronology violating spacetimes, including the ones considered in this paper, are multiply connected and in general, any multiply connected acausal spacetime should have a simply connected covering space with points identified under a discrete group of isometries. The effective action density will be given as a function of the metric parameters which determine the spacetime interval separating two of the identified points and in a thermodynamic context, one can see that these parameters are just thermodynamic intensive variables, which were interpreted as temperature and angular velocity in the rotating fluid model considered here. Similar parameters will also exist if one analytically continues from some Euclidean metric to obtain a simply connected acausal spacetime. Therefore, one should always be able to Legendre transform the effective action in order to calculate the density of states, which must be defined as a function of the thermodynamically conjugate extensive variables. The evidence presented in this paper suggests that the resulting density of states will tend to zero as the parameters are adjusted so as to introduce CTCs. Thus it appears that quantum mechanics naturally forbids acausal behaviour. There are no quantum states available for causality violating field configurations.

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